Signature of Invigilators

1. ....

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Roll No. | | | | | |

(In figures as in Admit Card)

MATHEMATICAL SCIENCES

Paper III

Roll No. ....

....

(In words)

J—0102

Name of Areas/Section (if any) ....

Time Allowed : $2\frac{1}{2}$ Hours] [Maximum Marks : 200

Instructions for the Candidates

1. Write your Roll number in the space provided on the top of this page.
2. Write name of your Elective/Section if any.
3. Answer to short answer/essay type questions are to be written in the space provided below each question or after the questions in test booklet itself. No additional sheets are to be used.
4. Read instructions given inside carefully.
5. Last page is attached at the end of the test booklet for rough work.
6. If you write your name or put any special mark on any part of the test booklet which may disclose in any way your identity, you will render yourself liable to disqualification.
7. Use of any calculator is prohibited.
8. There is no negative marking.
9. You should return the test booklet to the invigilator at the end of the examination and should not carry any paper outside the examination hall.

પરીક્ષાર્થીઓ માટેની સૂચનાઓ :

૧. આ પાનાની ટોચમાં દર્શાવેલી જગ્યામાં તમારો રોલ નંબર લખો.
૨. જો કોઈ વિકલ્પ/વિભાગ પસંદ કર્યા હોય તો તે યોગ્ય જગ્યાએ દર્શાવો.
૩. ટુંકા પ્રશ્નો/નિબંધ વિષેના જવાબો એ પ્રશ્નની નીચે અગર બાજુમાં આપેલી જગ્યામાં લખો. વધારાના કોઈ પાનાનો ઉપયોગ કરશો નહી.
૪. અંદર આપેલી સૂચનાઓ કાળજીપૂર્વક વાંચો .
૫. બુકલેટની પાછળ આપેલું છેલ્લુ પાનું રફ કામ માટે છે.
૬. બુકલેટ કોઈપણ ઠેકાણે તમારું નામ કે કોઈ ચોકકસ સંજ્ઞા કરવી નહીં કે જે તમારી ઓળખ પૂરી પાડે. આ તમને પરીક્ષા માટે ગેરલાયક ઠેરવશે.
૭. કેલ્કયુલેટર નો ઉપયોગ કરાશે નહી.
૮. નકારાત્મક માર્કીંગ નથી.
૯. પરીક્ષા સમય પૂરો થઈ ગયા પછી આ બુકલેટ જે તે નીરીક્ષકને સોંપી દેવી. કોઈપણ પેપર પરીક્ષા રૂમની બહાર લઈ જવું નહી.

FOR OFFICE USE ONLY
Marks Obtained

Question Number	Marks Obtained	Question Number	Marks Obtained	Question Number	Marks Obtained
1		26			
2		27			
3		28			
4		29			
5		30			
6		31			
7		32			
8		33			
9		34			
10		35			
11		36			
12		37			
13		38			
14		39			
15		40			
16					
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24					
25					

Total Marks Obtained.................................

Signature of the co-ordinator....................

(Evaluation)

MATHEMATICAL SCIENCES

PAPER III

Note :— (*i*) This paper contains *forty* **(40)** questions, each carrying **twenty (20)** marks. The first *twenty* **(20)** questions pertain to mathematics, the remaining to statistics.

(*ii*) Attempt any *ten* questions.

(*iii*) Answer each question in about **200** words (2 pages).

1. (*a*) Show that if f is monotonically increasing function on $[a, b]$, then it is bounded and Riemann integrable on $[a, b]$. Show that a monotonically increasing function on (a, b) need not be bounded.

(*b*) Show that if $-\infty < a < b < \infty$, then the closed interval $[a, b]$ is compact. Is $(0, 1)$ compact ? Justify your answer.

2. (*a*) Let

$$f(x, y) = \frac{x^3}{x^2 + y^2}, \quad (x, y) \neq (0, 0)$$
$$= 0, \quad (x, y) = (0, 0)$$

Show that :

(*i*) f is continuous in $\mathbf{R}^2$;

(*ii*) the directional derivative at $(0, 0)$ for $f(x, y)$ exists along every line through the origin;

(*iii*) f is not differentiable at $(0, 0)$.

(*b*) Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by

$$f(x, y) = (e^x \cos y, e^y \sin y)$$

and $\mathrm{E} = \{(x, y) \in \mathbf{R}^2 : 0 < y < 2\pi\}$

(*i*) Prove that f is not one-to-one on $\mathbf{R}^2$ but it is one to one when restricted to E.

(*ii*) What is $\mathrm{F} = f(\mathrm{E})$?

(*iii*) If $g : \mathrm{F} \to \mathrm{E}$ is the inverse function, find $\mathrm{D}g(0, 1)$.

3. Find all functions which are continuous on the unit disc $|z| < 1$ and for which $f(z) = f(z^2)$ for all z in the disc.

4. Prove that the maximum value of $|e^z|$ on and inside the closed square formed by the lines $x = \pm 1$, $y = \pm 1$ is e.

5. Suppose A and B are ideals of a commutative ring R. If [A : B] is defined by [A : B] = $\{r \mid r \in R$ such that $rb \in A$ for all $b \in B\}$, prove that [A : B] is an ideal of R.

6. Let $T : V \to V$ be a diagonalizable linear map. Show that the minimal polynomial of T has all its roots simple.

7. (*a*) If $\{x_n\}$ is a sequence of distinct points in a metric space converging to x, show that the set $\{x, x_1, x_2, x_3, ..., x_n, ...\}$ is compact.

(*b*) If f and g are real-valued Lebesgue measurable functions defined on R, show that $f + g$ and f^2 are Lebesgue measurable.

8. Determine all the 3-sylow subgroups of S_4. Select *two* of them and prove that they are conjugate.

9. (*a*) Let X be a normed linear space. If $\{x_n\}$, $\{y_n\}$ are sequences in X and $\{\alpha_n\}$ is a sequence of scalars, such that $x_n \to x$, $y_n \to y$ and $\alpha_n \to \alpha$, show that $x_n + y_n \to x + y$ and $\alpha_n x_n \to \alpha x$. Hence show that if Y is a subspace of X, then $\overline{Y}$ (closure of Y) is also a subspace of X.

(*b*) Show that the closed unit ball of a normed linear space, in general, need not be compact in norm topology. Give a condition under which it is compact.

10. (*a*) Define a closed map between two topological spaces.

Let X be compact and Y be Hausdorff. Show that any continuous map $f : X \to Y$ is a closed map.

(*b*) Let X be a topological space and Y be another topological space which is Hausdorff. Let $f : X \to Y$ be any continuous map and

$$\Gamma_f = \{(x, y) \in X \times Y \mid y = f(x)\}$$

be its graph. Show that Γ_f is a closed subset of X × Y.

11. Show that $\mathbf{R}^2 \setminus \{0\}$ is homotopically equivalent to the unit circle S^1 but not homeomorphic to it.

12. In each room of a house there are even number of doors leading to another room or to outside. Prove that the house has even number of exits (that is, even number of doors leading out of the house).

13. (a) Classify and reduce the following equation to a canonical form and solve it :

$$3\frac{\partial^2 u}{\partial x^2} + 10\frac{\partial^2 u}{\partial x \partial y} + 3\frac{\partial^2 u}{\partial y^2} = 0.$$

(b) Show that the eigenfunctions belonging to two different eigenvalues are orthogonal with respect to the weight function $r(x)$ in the interval (a, b) for the system :

$$(p(x)y')' + [q(x) + \lambda r(x)]y = 0,$$

$$y(a) = 0, \; y(b) = 0,$$

where $p(x)$, $q(x)$ and $r(x)$ are continuous functions on (a, b) with $r(x) \geq 0$.

14. For a positive integer n of the type $4k + 3$, let $d_1(n)$ be the number of divisors of n of the type $4m + 1$ and $d_3(n)$ be the number of divisors of n of the type $4m + 3$. Prove that $d_1(n) = d_3(n)$.

15. Derive Lagrange's equations for a holonomic dynamical system in the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \; ; \quad j = 1, 2, \ldots\ldots n,$$

where T is the kinetic energy of the system at time t when the system is specified by the n generalized coordinates $q_j (j = 1, 2, 3, \ldots\ldots., n)$ and Q_j are the generalized forces.

16. (a) Determine the principal directions and principal strains for the strain tensor $E = (E_{ij})$, given by

$$E = \begin{pmatrix} e & e & e \\ e & e & e \\ e & e & e \end{pmatrix}.$$

(b) Prove that for an incompressible fluid, the motion is possible when velocities at a point (x_1, x_2, x_3) are given by $v_1 = \frac{3x_1x_3}{r^5}$, $v_2 = \frac{3x_2x_3}{r^5}$, $v_3 = \frac{3{x_3}^2 - r^2}{r^5}$, where $r^2 = {x_1}^2 + {x_2}^2 + {x_3}^2$. Is this motion irrotational ? If yes, then find the velocity potential.

17. Let α be a smooth regular curve in $\mathbf{R}^3$ which lies on the unit sphere $S^2 = \{x^2 + y^2 + z^2 = 1\}$. Show that the curvature of α at each point is at least 1 and it is equal to 1 everywhere if and only if α lies along a great circle.

18. (a) Use variational method to obtain the function $y = y(x)$, which extremizes the integral $\int_0^1 y'^2 \, dx$ subject to the end conditions $y(0) = 0$ and $y(1) = 0$ and the constraint $\int_0^1 y dx = 1$.

(b) Show that the area of the surface of revolution of a curve $y = y(x)$ is

$$2\pi \int_{x_1}^{x_2} y(1 + y'^2)^{1/2} \, dx.$$

Hence show that for this to be a minimum, the curve must be a catenary.

19. (a) If $\lim_{t \to 0} \frac{f(t)}{t}$ exists and $L\{f(t)\} = F(s)$, show that the Laplace transform of $\frac{f(t)}{t}$ is $\int_s^\infty F(u) \, du$.

(b) Solve the following integral equation, using the Fourier integrals,

$$\int_0^\infty f(x) \cos \alpha x \, dx = \begin{cases} 1 - \alpha, & 0 \le \alpha \le 1 \\ 0 \;\;, & \alpha > 1 \end{cases},$$

and show that $\int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$.

20. (a) Let $x = s$ be a solution of $x = g(x)$ and suppose $g(x)$ has a continuous derivative in some interval J containing s. Then if $| \, g'(x) \, | \le k < 1$ in J, show that the iteration process defined by $x_{n+1} = g(x_n)$, $n = 0, 1, 2, \ldots\ldots\ldots$ converges for any x_0 in J. Apply this result to find a solution of $f(x) = x^3 + x - 1$ correct to two decimal digits.

(b) If λ be an eigenvalue of an arbitrary $n \times n$ matrix $A = (a_{ij})$, then for some integer i, $1 \le i \le n$, we have :

$|a_{ii} - \lambda| \le |a_{i1}| + |a_{i2}| + \ldots + |a_{i,i-1}| + |a_{i,i+1}| + \ldots + |a_{in}|$.

Use this result to show that the spectrum of the matrix

$$A = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 5 & 1 \\ 0.5 & 1 & 1 \end{pmatrix}$$

must lie in the intervals $[-1, 2.5]$ and $[3.5, 6.5]$ on the real axis.

21. (a) Discuss briefly the weaknesses of the simplex method. How is it revised to overcome them ?

(b) Describe Wolfe's algorithm used to solve a quadratic programming problem.

22. (a) Discuss in brief the advantages and weaknesses of the dual simplex method. Also write various criteria of the method and their roles.

(b) Use dual simplex method to

$$\begin{aligned} \text{Min.} \quad & z = 3x_1 + 2x_2 \\ \text{s.t.} \quad & 3x_1 + x_2 \ge 3 \\ & 4x_1 + 3x_2 \ge 6 \\ & x_1 + x_2 \le 3 \\ & x_1, \ x_2 \ge 0 \end{aligned}$$

23. (a) State Borel zero-one law.

(b) Let $\{X_n\}$ be a sequence of i.i.d. exponential r.v.s with $EX_n = 1$. Let $A_n = \{X_n > a_n\}$ and $B_n = \{X_n > b_n\}$. Find conditions on a_n and b_n such that

$$P(\limsup A_n) = 0 \text{ and } P(\limsup B_n) = 1.$$

24. (*a*) State Liapounov's central limit theorem.

(*b*) Let $\{X_n\}$ be a sequence of independent r.v.s. where X_n follows Bernoulli with success probability p_n, $n \geq 1$. Show that Liapounov's condition is satisfied if $\sum_{n=1}^{\infty} p_n q_n$ diverges, where $q_n = 1 - p_n$.

25. Let $\{X_n\}$ be a sequence of i.i.d. Bernoulli random variables with $P(X_n = 1) = p$, and N be a Poisson random variable, independent of X_i's, with parameter λ. Find the probability distribution of $S_N = \sum_{i=1}^{N} X_i$ $(S_0 \equiv 0)$.

26. Decompose the following distribution function (d.f.) F as $\alpha F_1 + (1 - \alpha)F_2$, where F_1 is the d.f. of a discrete r.v. and F_2 is a continuous d.f.

$$F(x) = \begin{cases} 0 & \text{if } x < -2 \\ \dfrac{x+3}{4} & \text{if } -2 \leq x < 0 \\ 4/5 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

27. (*a*) State an expression for the *r*th moment of a r.v. X in terms of its characteristic function.

(*b*) Find $E(X^{2n})$ for the standard normal r.v. using the relation stated in (*a*).

28. Let $(X_1, X_2, \ldots\ldots X_n)$, $n \geq 2$ be i.i.d. Bernoulli with $P(X_i = 1) = \theta$. Let

$$T_1(X_1, X_2) = \begin{cases} 1 & \text{if } X_1 = 1, X_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

Show that $T_1(X_1, X_2)$ is an unbiased estimator of $\theta(1 - \theta)$. Obtain Rao-Blackwellized version of T_1 with respect to minimal sufficient statistic $T = \sum_{1}^{n} x_i$ and show that it is UMVUE of $\theta(1 - \theta)$.

29. (*a*) Define Decision problem, Bayes risk with respect to a given prior and Bayes estimator.

(*b*) Using a quadratic loss function, obtain Bayes rule for estimating θ when

$$f(x, \theta) = \frac{e^{-\theta}\theta^x}{x!} \qquad \theta > 0, \; x = 0, 1, 2, ...$$

and the prior distribution of θ is given by

$$f(\theta) = \frac{1}{\beta^\alpha \Gamma\alpha} e^{-\theta/\beta} \theta^{\alpha-1} \quad \theta > 0, \; \alpha, \beta > 0.$$

30. Define most powerful test (m.p.t.). Show that m.p.t. of size α $(0 \leq \alpha < 1)$ always exists for testing a simple null hypothesis against simple alternative.

31. Discuss the independence of two attribute in $r \times c$ contingency table. What is Karl Pearson's coefficient of contingency ? How does it help us to determine the association between two attributes ?

A serum supposed to have effect in preventing cold was tested on 50 individuals and their records for one year were compared with the records of 50 individuals which were not treated, and they are given below :

	No cold	Suffered from cold	Total
Treated	25	25	50
Untreated	22	28	50
Total	47	53	100

Compute the coefficient of contingency by Karl Pearson and comment on the same. (Chi-square value at 0.05 level for 1 d.f. is 3.84).

32. Let $X = (X_1, X_2, X_3)' \sim N_3(0, \Sigma)$, where

$$\Sigma = \begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{pmatrix}$$

Find the value of ρ for which $X_1 + X_2 + X_3$ and $X_1 - X_2 - X_3$ are independent.

33. (a) Explain what you understand by Gauss Markoff's set-up $(Y, X_{\beta}, \sigma^2 I)$.

(b) (i) Consider linear functions $L'Y$ and $M'Y$ and find covariance between them.

(ii) If $P'\beta$ is estimable, show that the variance of $P'\hat{\beta}$ is minimum in the class of linear unbiased estimators of $P'\beta$.

34. What is cluster sampling ? When is it needed ? Suggest an Unbiased estimator of population mean per unit based on a SRSWOR sample of n clusters of equal size of M units from a population of N clusters. Obtain its variance in terms of intra class correlation coefficient and deduce the condition when it is better than SRSWOR mean per unit estimator.

35. The following is the layout of some block design with $b = 7$ blocks, $v = 7$ treatments denoted by A – G.

B_1 . B , C , F	B_5 . B , E , G
B_2 . A , C , E	B_6 . A , F , G
B_3 . A , B , D	B_7 . D , E , F.
B_4 . C , D , G	

(a) Identify the design and write its parameters.

(b) Verify whether the design is connected, balanced, orthogonal. Provide arguments in justification of your answer.

36. (a) Define a Weakly Stationary Stochastic Process.

(b) Examine whether the two stochastic processes $\{X_n\}$ and $\{Y_n\}$ given below are Weakly Stationary :

(i) $X_n = U \cos n\theta + V \sin n\theta$, where U and V are uncorrelated r.v.s each with mean 0 and variance 1 and $\theta \in [-\pi, \pi]$.

(ii) $\{Y_n\}$ is a Poisson process.

37. Find $\lim_{n \to \infty} P^n$, where

$$P = \begin{pmatrix} 0 & 2/3 & 1/3 \\ 3/8 & 1/8 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

38. Customers arrive at a service counter according to the Poisson process with a mean arrival rate of 5 customers per hour. Find :

(i) P (no customer arrives in a time gap of 10 minutes)

(ii) P (Not more than 2 customers arrive in a time gap of 10 minutes)

(iii) P (Not more than one customer arrived in the first half hour given that exactly four customers arrived in the first hour)

(iv) P (2 customers arrived by the end of first half hour given that the 1st customer arrived at the end of first 20 minutes).

39. What do you understand by control chart for attributes ? Discuss P-chart and C-chart and explain the situations in which these are used.

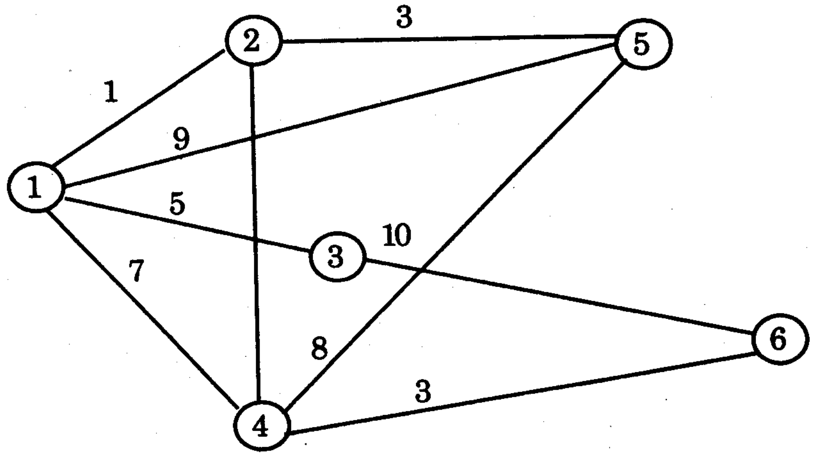
Draw a suitable control chart for the following data pertaining to the number of coloured thread (considered as defects) in 15 pieces of cloth (in a certain number of synthetic fibre) and state your conclusions.

7, 12, 3, 20, 21, 5, 4, 3, 10, 8, 0, 9, 6, 7, 20.

40. (a) List various network optimization algorithms of the network models and real life situations for their applications.

(b) A cable company is in the process of providing cable service to five new housing development areas of Vadodara.

The figure given below depicts the potential TV linkages among the five areas. The cable kilometers are shown on each branch. Determine the most economical cable network.



Q. No. ....